Limits and Continuity

A Short Review

Lecture Notes

September 15, 2026

Same as for $\mathbb{R}^2$.

1 Limits

Definition (limit)

Let f be a function defined on a set $K \subset \mathbb{C}$. We say f has a **limit** A as $z \to z_0$ if

$$\forall \varepsilon > 0 \; \exists \delta > 0 : 0 < |z - z_0| < \delta, \; z \in K \implies |f(z) - A| < \varepsilon.$$

Properties

1. If the limit exists, it is unique, provided z_0 is a limit point of K (i.e. $\forall \delta > 0$: $B(z_0, \delta) \cap K \setminus \{z_0\} \neq \varnothing$).
2. $\lim_{z \to z_0} [f(z) + g(z)] = \lim_{z \to z_0} f(z) + \lim_{z \to z_0} g(z)$ (if both exist).
3. $\lim_{z \to z_0} [f(z) \cdot g(z)] = \lim_{z \to z_0} f(z) \cdot \lim_{z \to z_0} g(z)$ (if both exist).
4. $\lim_{z \to z_0} \dfrac{f(z)}{g(z)} = \dfrac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)}$ (if both exist and $\lim_{z \to z_0} g(z) \neq 0$).
5. $\lim_{z \to z_0} f(z) = A \iff \lim_{z \to z_0} \operatorname{Re} f(z) = \operatorname{Re} A$ and $\lim_{z \to z_0} \operatorname{Im} f(z) = \operatorname{Im} A$.
6. $\lim_{z \to z_0} f(z) = A \implies \lim_{z \to z_0} \overline{f(z)} = \overline{A}$.

Proof idea.

- (1), (2), (3), (4) – as in the real case.
- (6) Follows from $|f(z) - A| = |\overline{f(z)} - \overline{A}| < \varepsilon$.
- (5) $(\Leftarrow)$ $f = \operatorname{Re} f(z) + i \operatorname{Im} f(z)$; use (2). $(\Rightarrow)$ $\operatorname{Re} f(z) = \frac{1}{2}(f(z) + \overline{f(z)})$ and $\operatorname{Im} f(z) = -\frac{i}{2}(f(z) - \overline{f(z)})$; use (6) and (2).

Important Property (restriction)

Let $K_1, K_2 \subset K$, and suppose $\lim\limits_{\substack{z \to z_0 \\ z \in K}} f(z) = A$. Then

$$\lim_{\substack{z \to z_0 \\ z \in K_1}} f(z) = \lim_{\substack{z \to z_0 \\ z \in K_2}} f(z) = A.$$

Proof. $\forall \varepsilon > 0\ \exists \delta > 0$: $0 < |z - z_0| < \delta \implies |f(z) - A| < \varepsilon$; in particular this holds for $z \in K_1 \subset K$. $\square$

Corollary

Let $K_1, K_2 \subset K$. If $\lim\limits_{\substack{z\to z_0\\ z\in K_1}} f(z) \neq \lim\limits_{\substack{z\to z_0\\ z\in K_2}} f(z)$, then $\lim\limits_{\substack{z\to z_0\\ z\in K}} f(z)$ does not exist.

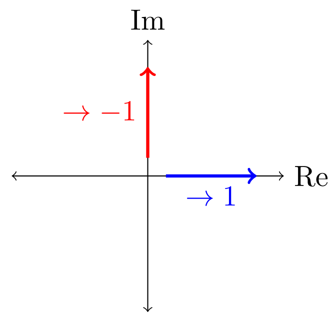
Warning: the converse fails. Agreement of the limit along every *straight* path through z_0 does *not* by itself guarantee that $\lim_{z\to z_0} f(z)$ exists – a counterexample typically needs a curved path (e.g. a parabola) along which f tends to a different value.

2 Example: $\bar{z}/z$ Has No Limit at 0

Claim. $\lim\limits_{z\to 0} \dfrac{\bar{z}}{z}$ **does not exist**.

On the real axis ($z = x$): $\lim\limits_{x\to 0} \dfrac{x}{x} = 1$.

On the imaginary axis ($z = iy$): $\lim\limits_{y\to 0} \dfrac{-iy}{iy} = -1$.



Alternatively, using polar coordinates $z = |z| \operatorname{cis}(\varphi)$:

$$\frac{\bar{z}}{z} = \frac{|z| \operatorname{cis}(-\varphi)}{|z| \operatorname{cis}(\varphi)} = \operatorname{cis}(-2\varphi).$$

This shows we get **different limits along different rays**!

3 Continuity

As usual: f is **continuous** at z_0 if $\lim\limits_{z\to z_0} f(z) = f(z_0)$.

4 Limits Involving Infinity

Definitions (limits involving ∞)

All of this can be done for the limit equal to ∞, but we need to use the **spherical metric**:

$$\lim_{z\to z_0} f(z) = \infty \iff \lim_{z\to z_0} d(f(z), \infty) = 0 \iff \lim_{z\to z_0} \frac{2}{\sqrt{1 + |f(z)|^2}} = 0 \iff \lim_{z\to z_0} \frac{1}{|f(z)|} = 0.$$

Defining the limit at infinity:

$$\begin{aligned}\lim_{z\to\infty} f(z) = A &\iff \forall \varepsilon > 0\ \exists \delta > 0 : d(z, \infty) < \delta \implies |f(z) - A| < \varepsilon \\ &\iff \forall \varepsilon > 0\ \exists \delta > 0 : \frac{1}{|z|} < \delta \implies |f(z) - A| < \varepsilon \\ &\iff \forall \varepsilon > 0\ \exists R > 0 : |z| > R \implies |f(z) - A| < \varepsilon \qquad \left(\text{where } R = \tfrac{1}{\delta}\right).\end{aligned}$$

Important (and easy) observation. If $z_0 \neq \infty$, then

$$\lim_{z \to z_0} |z - z_0| = 0 \iff \lim_{z \to z_0} d(z, z_0) = \lim_{z \to z_0} \frac{2|z - z_0|}{\sqrt{1 + |z|^2}\sqrt{1 + |z_0|^2}} = 0.$$